Crossover between Ballistic and Normal Diffusion

Syuji Miyazaki

Department of Applied Analysis and Complex Dynamical Systems, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan

(Received June 23, 2005)

Crossover between ballistic motion and normal diffusion is studied based on the continuous-time random walk (CTRW) approach in order to analyze universal properties of strongly correlated motion and the decay process of correlation in deterministic diffusion. There exists a characteristic time scale τ. For the time region $t \ll \tau$, ballistic motion is observed, which is followed by normal diffusion for $t \gg \tau$. Higher-order moments are analytically obtained, and it is found that they obey scaling relations that are reminiscent of the generalized extended self-similarity (GESS) found in turbulent systems. As a simple dynamical system for numerical simulations, the climbing sine map in the vicinity of band crisis is considered. Good agreement between the theory and the numerical simulations is observed.

§1. Introduction

From the viewpoint of deterministic diffusion,[1] diffusion is caused by chaotic dynamics in a dynamical system. The invariant sets relevant to the chaotic dynamics in the phase space suffer bifurcations when the control parameter is changed. Long correlations occur in the vicinity of the bifurcation point, leading to anomalous diffusion. There exists a characteristic time τ, which corresponds to the mean free time in the case of the diffusion caused by thermal noise. Unlike the mean free time, this characteristic time τ may diverge in the vicinity of the bifurcation point. Tangent bifurcation is an example. Thus, it is important to characterize crossover phenomena between anomalous and normal diffusion observed respectively for $t \ll \tau$ and for $t \gg \tau$ by use of various scaling properties, as is also the case for turbulent phenomena.[2] We attempted to find scaling laws that hold from the anomalous subdiffusion region into the normal diffusion region as a whole, and compare them with generalized scaling laws, like generalized extended self-similarity (GESS),[3] which were introduced to describe turbulence at intermediate Reynolds numbers. Miyazaki et al. succeeded in finding such scaling laws related to modulational intermittency[4)–7] and to superdiffusion in oscillating convection flows.[8]

§2. Crossover between ballistic motion and normal diffusion

We derive the crossover between anomalous ballistic motion and normal diffusion by using the CTRW velocity model,[9] which describes motion consisting of uniform motion and instantaneous changes of direction. Let $\psi(t)$ be the probability density function (PDF) to go straight in one direction up to time t, 'flight duration', which emerges as a straight segment in Fig. 1. We assume the following PDF

$$\psi(t) = \frac{1}{\tau}\exp\left(-\frac{t}{\tau}\right), \tag{2·1}$$

where τ corresponds to a characteristic time scale of the crossover, which is equal to the average flight duration. Based on the CTRW velocity model, we obtain the mean square displacement corresponding to $\psi(t)$ given by Eq. (2·1) as

$$\frac{\langle r^2\rangle(t)}{2Dt} = \bar{\phi}\left(\frac{t}{2\tau}\right), \tag{2·2}$$

with the diffusion constant $D = \tau v^2$, and the scaling function

$$\bar{\phi}(z) = 1 - \frac{1}{2z}\left(1 - \exp(-2z)\right). \tag{2·3}$$

We have $\bar{\phi}(z) \sim z$ for $z \ll 1$, and $\bar{\phi}(z) \sim 1$ for $z \gg 1$. From the CTRW velocity model and saddle-point calculations, we can also obtain for the $2m$-th moment as $\langle r^{2m}\rangle(t) = N_m\, g\left(\frac{t}{2m\tau}\right)\ \left[\hat{t}\left(\frac{t}{2m\tau}\right)\right]^m$ with $N_m = \Gamma(2m+1)(2e\tau^2v^2)^m$, $\phi(z) = \frac{\sqrt{1+z^2}-1}{z}\exp(-z + \sqrt{1+z^2})$, $g(z) = \frac{z+1+\sqrt{1+z^2}}{\sqrt{1+z^2+\sqrt{1+z^2}}}$, and $\hat{t}(z) = z\,\phi(z)$. When we consider the moment normalized by the function g

$$\frac{\langle r^{2m}\rangle(t_m)}{g\left(\frac{t}{2\tau}\right)} \propto \left\{\frac{\langle r^{2n}\rangle(t_n)}{g\left(\frac{t}{2\tau}\right)}\right\}^{m/n}, \tag{2·4}$$

which holds for all t. This scaling relation is an analog of GESS.[3)]

Now we compare the theory with numerical simulations using a simple dynamical system on the real line called the climbing sine map: $x_{n+1} = x_n + a\sin(2\pi x_n)\quad (a > 0)$.[10),11)] Diffusive motion starts at $a = 0.7326441325\ldots$, below which only nondiffusive motion is observed. There exist cascades of tangent bifurcations and band crises in this system. In the vicinity of the bifurcation points, diffusive motion is so strongly correlated that the diffusion constants are either vanishing or diverging.[10)]

Let us consider the bifurcation point $a = a_c = 1.1082300133\ldots$, below which only the running chaotic solution to the right or the left is observed. The direction of motion depends on the initial value x_0. There exist two attractors corresponding to leftward and rightward motions. These two attractors (bands) collapse and a wider attractor is simultaneously born at the bifurcation point called band crisis. The situation is similar to that of the three-band crisis at the right edge of the period-3 window of the bifurcation diagram of the logistic map. Strong correlations of the local expansion rate (local Lyapunov exponent) in the vicinity of the band crisis were analyzed previously.[12)–14)] This time we bring the strong velocity correlations of diffusive motion into the question. Figures 1 and 2 depict time series of position x_n and velocity $v_n = x_{n+1} - x_n$ at $a = a_c(1+\epsilon)$ with $\epsilon = 0.00002$ just after the band crisis.

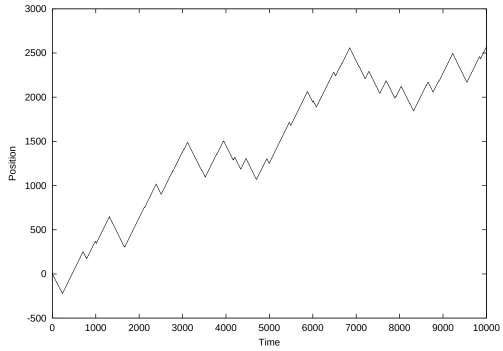

Fig. 1. Time series of position x_n at $a = a_c(1+\epsilon)$ with $\epsilon = 0.00002$ just after the band crisis.

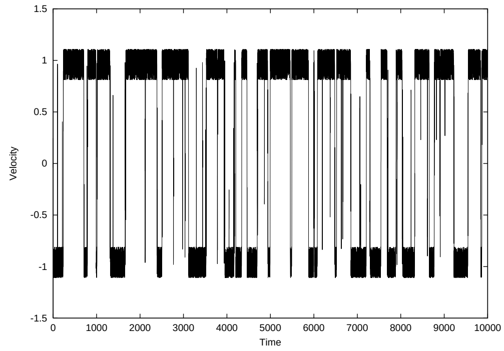

Fig. 2. Time series of velocity $v_n = x_{n+1} - x_n$ at $a = a_c(1+\epsilon)$ with $\epsilon = 0.00002$ just after the band crisis.

It has already been shown that the distribution of the duration of one-directional motion of the system is given by an exponential function (2·1),[10] so that the statistical properties described in the preceding section are valid for the climbing sine map just after the band crisis. In Fig. 3, the scaling function of the second moment (2·2) for $\epsilon = 0.02(+), 0.002(\times), 0.0002(*)$ is plotted against the scaled time $t/(2\tau)$, where τ is the mean duration of one-directional motion. The theory given by (2·3) is also drawn as a solid line. In Fig. 4, the compensated $2m$-th moments $\langle r^{2m}\rangle(t_m)/g(\frac{t}{2\tau})$ at moment-order dependent scaled time $t = t_m = mt$ are plotted against the compensated second moment $\langle r^2\rangle(t_1)/g(\frac{t}{2\tau})$ for $m = 2(+), 3(\times), 4(*)$ and $5(\Box)$. The theory given by (2·4) is also drawn as lines. Good agreement is observed in both Figs. 3 and 4.

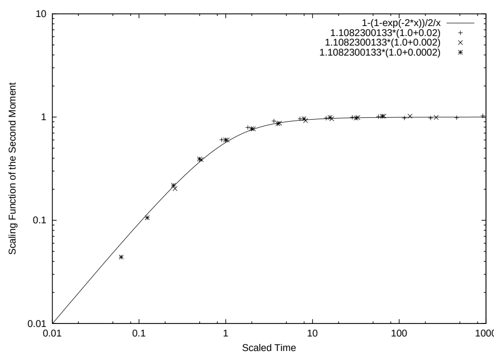

Fig. 3. The scaling function of the second moment (2·2) for $\epsilon = 0.02(+), 0.002(\times)$, $0.0002(*)$ is plotted against the scaled time t/τ, where τ is the mean duration of one-directional motion. The theory given by (2·3) is also drawn as solid line.

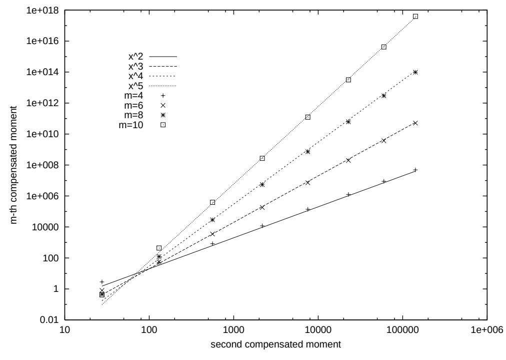

Fig. 4. The compensated $2m$-th moments $\langle r^{2m}\rangle(t_m)/g(\frac{t}{2\tau})$ at moment-order dependent scaled time $t = t_m = mt$ are plotted against the compensated second moment $\langle r^2\rangle(t_1)/g(\frac{t}{2\tau})$ for $m = 2(+), 3(\times), 4(*)$ and $5(\Box)$. The theory given by (2·4) is also drawn as lines.

§3. Concluding remarks

A test particle under the influence of deterministic diffusion has strongly correlated motion for the time scale which is much shorter than a characteristic time estimated from the correlation time due to the underlying chaotic dynamics causing deterministic diffusion. General nonhyperbolic dynamical systems found in the realistic world have rich structures of bifurcations. The bifurcation diagram of the logistic map illustrates this situation most clearly. In the vicinity of bifurcation points, the above characteristic time becomes so long that it is important to analyze the universal properties of strongly correlated motion and the decay process of correlation in deterministic diffusion. Based on this idea, the scaling properties of higher order moments were derived for the simple system describing crossover between ballistic motion and normal diffusion. Numerical simulations using the climbing sine map in the vicinity of the band crisis agree with our theory very well.

Acknowledgements

This study was supported by a Grant-in-Aid for the 21st Century COE "Center of Excellence for Research and Education on Complex Functional Mechanical Systems" from the Ministry of Education, Culture, Sports, Science and Technology.

References

1) P. Gaspard, *Chaos, Scattering and Statistical Mechanics* (Cambridge University Press, Cambridge, 1998).
2) A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics* (MIT Press, Cambridge, MA, 1975).
3) R. Benzi, L. Biferale, S. Ciliberto, M. V. Struglia and R. Tripiccione, Physica D **96** (1996), 162.
4) S. Miyazaki, T. Harada and A. Budiyono, Prog. Theor. Phys. **106** (2001), 1051.
5) S. Miyazaki and K. Ito, Prog. Theor. Phys. **108** (2002), 999.
6) N. Tsukamoto, S. Miyazaki and H. Fujisaka, Phys. Rev. E **67** (2003), 016212.
7) S. Miyazaki, T. Harada, K. Ito and A. Budiyono, *Recent Research Developments in Physics* **4** (Transworld Research Network, Kerala, India, 2003), p. 155.
8) K. Ito and S. Miyazaki, Prog. Theor. Phys. **110** (2003), 875.
9) G. Zumofen and J. Klafter, Phys. Rev. E **47** (1993), 851.
10) M. Schell, S. Fraser and R. Kapral, Phys. Rev. A **26** (1982), 504.
11) N. Korabel and R. Klages, Phys. Rev. Lett. **89** (2002), 214102.
12) S. Miyazaki, N. Mori, T. Yoshida, H. Mori, H. Hata and T. Horita, Prog. Theor. Phys. **82** (1989), 863.
13) T. Yoshida, S. Miyazaki, H. Mori, T. Kobayashi, T. Horita and H. Hata, Prog. Theor. Phys. **82** (1989), 879.
14) T. Yoshida and S. Miyazaki, Prog. Theor. Phys. Suppl. No. 99 (1989), 64.